The rotation distance between two binary rooted trees

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Young Geometric Group Theory X

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Rotation

A *rotation* in a binary tree is a local restructuring of the tree, executed by collapsing an internal edge of the tree to a point, thereby obtaining a node with three children, and then re-expanding the node of order three in the alternative way.

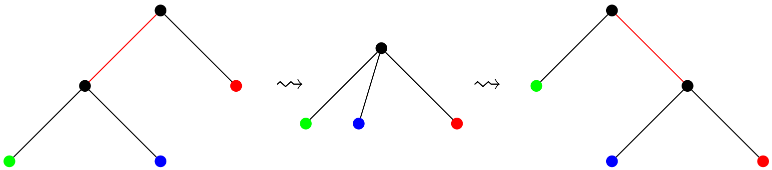

Figure 1: A rotation

The Rotation Distance Problem

Rotation Distance

The *rotation distance* between two binary rooted trees with the same number of nodes is the minimum number of rotations needed to convert one tree into another.

- What is the maximum rotation distance between any pair of n-node binary trees?
- Is there a polynomial time algorithm (in the number of nodes of the trees) to determine the rotation distance between a given pair of trees?

History

- **Motivated by binary search trees:** Rotations provide a simple mechanism for "balancing" binary search trees – the efficiency of storing and retrieving information from binary search trees depends on their height and balance.
- **Originally mentioned:** Čulik, Wood 1982
- **Other work:** Thurston 1988; Dehornoy 2010; Cleary, Pallo

Thompson's group F

Thompson's Group F

The group of piecewise linear homeomorphisms of the unit interval $[0, 1]$, which are differentiable except at finitely many dyadic rationals, and at the intervals of differentiability the derivatives are powers of 2.

- Originally defined by Richard Thompson in 1965, alongside T and V.
- T and V are finitely-presented infinite simple groups and F is a finitely-presented group with a simple commutator subgroup.

Thompson's Group F

A partition of the unit interval by dyadic rationals can be denoted by a binary rooted tree.

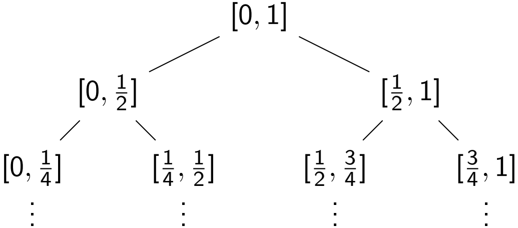

Figure 2: A binary rooted tree

An element of F can be associated to two binary rooted trees with the same number of nodes.

Link with rotation distance

- **Dehornoy 2010:** "On the rotation distance between binary trees". Link between a particular presentation of F and rotation distance.
- **Bleak, Quick, K 20??:** Independently produced this presentation using *rearrangement groups of fractals* (Belk, Forrest 2019).

A presentation for Thompson's group F

Our presentation for Thompson's group F is:

$$F = \langle \mathcal{X} \mid \mathcal{R} \rangle$$

The generating set $\mathcal{X}$ is

$$\mathcal{X} = \{ f_\alpha \mid \alpha \in \{0,1\}^* \},$$

where f_α acts as follows on points in $[0,1]$ with the prefix $\alpha = e_1 \dots e_n \in \{0,1\}^*$, and as the identity homeomorphism on the rest of the interval:

$$([\alpha e_{n+1} e_{n+2} \dots]) f_\alpha = \begin{cases} [\alpha 0 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1} e_{n+2} = 00, \\ [\alpha 10 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1} e_{n+2} = 01, \\ [\alpha 11 e_{n+2} e_{n+3} \dots] & \text{if } e_{n+1} = 1. \end{cases}$$

A presentation for Thompson's group F

The map f_α is illustrated in the following diagram (*rectangle/Thurston diagram*):

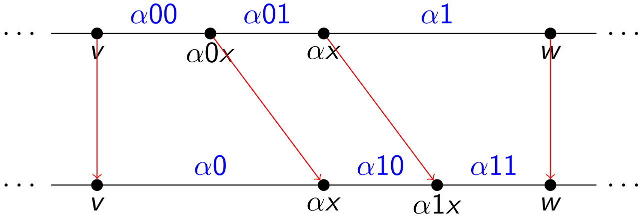

Figure 3: The map f_α

For each $\beta \in \{0,1\}^*$, the point βx denotes the dyadic rational represented by $\beta 0\bar{1}$ and $\beta 1\bar{0}$.

The map f_α is illustrated in the following diagram (*tree pair diagram*):

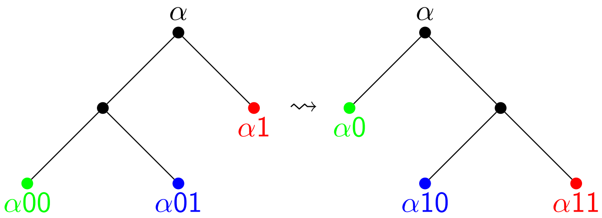

Figure 4: The map f_α

A presentation for Thompson's group F

The set of relations $\mathcal{R}$ is

$$\begin{aligned}\mathcal{R} = \big\{ R1 : &\quad {f_\beta}^{f_\alpha} = f_\beta \text{ for } \alpha \perp \beta, \\ R2 : &\quad {f_{\alpha 0}}^{f_\alpha} = f_\alpha {f_{\alpha 1}}^{-1}, \\ R3 : &\quad {f_{\alpha 00\gamma}}^{f_\alpha} = f_{\alpha 0\gamma}, \\ R4 : &\quad {f_{\alpha 01\gamma}}^{f_\alpha} = f_{\alpha 10\gamma}, \\ R5 : &\quad {f_{\alpha 1\gamma}}^{f_\alpha} = f_{\alpha 11\gamma} \big\},\end{aligned}$$

(for some $\alpha, \beta, \gamma \in \{0,1\}^*$).

Link with rotation distance

Theorem (Dehornoy 2010)

The rotation distance between two binary rooted trees with the same number of nodes is equal to the *length* of the corresponding element of F in terms of the generating set $\mathcal{X}$.

Sketch of proof

We observe that each map f_α acts similarly on a binary rooted tree as a rotation. The result follows.

Link with rotation distance

A rotation.

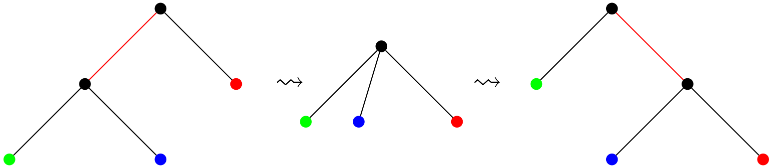

Figure 5: A rotation

An element of F (*tree pair diagram*).

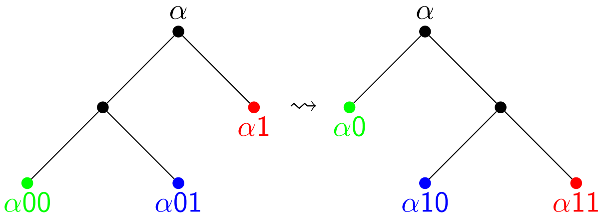

Figure 6: The map f_α

Current work

Result (Bleak, Quick, K)

A combinatorial algorithm which finds the length of an element of F in terms of the generating set $\mathcal{X}$.

Conjecture

Our combinatorial algorithm (which currently runs on exponential time) can be improved to run on polynomial time (or at least polynomial expected time).

Thank you!